3. Duality Theory in Infinite Horizon Optimization Models

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3.1 Introduction

In intertemporal resource allocation problems with no terminal date, price systems which characterize efficient or optimal allocations have figured prominently since the pioneering contribution by Malinvaud (1953). The method of duality theory that has been developed to study such problems relies on convex analysis and may be viewed as an extension of the corresponding theory for static or finite horizon allocation problems. The purpose of this survey is to introduce the reader to this method by showing how it has been applied in the literature dealing with optimal intertemporal allocation, when future utilities are discounted, which constitutes only a part (although a significant one) of the class of problems referred to above.

A major accomplishment of this literature is the result that, in a very general framework of capital accumulation (often referred to in the literature as a reduced-form model), optimal programs may be characterized by the existence of dual variables, interpreted as "shadow prices", such that at these prices the given program satisfies the so-called "competitive conditions" and the "transversality condition". The competitive conditions are analogous to those in static or finite horizon optimality problems, and involve myopic (generalized) intertemporal profit maximization. The fundamental difference stems from the infinite-horizon nature of the problem, and is captured by the transversality condition.

The usefulness of this central result may be described as follows. Sufficient conditions (in terms of shadow prices) for a program to be optimal can be used to check whether a candidate program is optimal, if one has a good idea of

[1] Discussion over the years with many persons has influenced my understanding of the subject matter covered in this essay. They include David Cass, Swapan Dasgupta, Ali Khan, Mukul Majumdar, Lionel McKenzie, Kazuo Nishimura, Bezalel Peleg, Debraj Ray and Itzhak Zilcha.

shadow prices that support such a program. This makes duality theory a principal alternative to dynamic programming methods in solving for an optimal program. Necessary conditions (in terms of shadow prices) for a program to be optimal can be used to obtain qualitative properties of an optimal program without necessarily solving for an optimal program.

Even though the theory of optimal growth dates back to the seminal contribution of Ramsey (1928), versions of the "price chracterization result", referred to above, were developed almost forty years later, in the papers of Gale (1967), McFadden (1967) and McKenzie (1968). Following Ramsey's lead, the principal concern of these papers was the theory of undiscounted optimal growth in general capital accumulation models. Subsequently, methods of duality theory were applied to the discounted case by Peleg (1970) and Peleg and Ryder (1972). However, it is only with the contribution of Weitzman (1973) that we have a completely satisfactory price characterization result for the discounted case. The setting for his result is a very general and flexible framework of capital accumulation (described here in Section 3.2), and his approach (combining elements of duality theory and dynamic programming) makes the logic of the result (and the assumptions needed for its validity) entirely transparent. We present the basic characterization result, following his approach, in Section 3.3.

Dual variables have been used very effectively in the literature on optimal intertemporal allocation in obtaining another major result, namely the existence of a non-trivial stationary optimal program, supported by "quasi-stationary" shadow prices. Versions of this result appear in Sutherland (1970) and Peleg and Ryder (1974). But, for the general framework described in Section 3.2, the result was developed later by Flynn (1980) and McKenzie (1982). The approach used in these two papers is to establish the existence of a discounted golden-rule (analogous to a golden-rule in the undiscounted case) by a fixed point argument, and then support this discounted golden-rule by appropriate dual variables. We present this theory in Section 3.4.

The fact that there exists a stationary optimal program with quasi-stationary price support allows one to revisit the basic price characterization result (of Section 3.3), and develop an alternative version of it which helps to identify non-optimal competitive programs in a finite number of periods. The transversality condition is an asymptotic condition, and can never be verified in finite time. It turns out that a convenient period-by-period condition can replace the transversality condition in the price characterization theorems, and so a violation of this condition in any period immediately signals non-optimality. Such a period-by-period condition was first proposed and established by Brock and Majumdar (1988) in the undiscounted case, and the theory for the discounted case was developed subsequently in Dasgupta and Mitra (1988). We present this theory in Section 3.5.

Although the transversality condition is both necessary and sufficient for optimality of competitive programs, there is a fairly wide and interesting class of models in which the competitive conditions alone are sufficient to ensure optimality, and the transversality condition is superfluous. That is, programs which are competitive necessarily satisfy the transversality condition and are therefore optimal. These are models which satisfy a "reachability" property, introduced by Dasgupta and Mitra (1999a). Thus, the competitive and transversality conditions are independent restrictions only in models where the reachability property is violated. This finding is presented in Section 3.6.

A framework of optimal growth that has received considerable attention in the literature is one in which utility is derived from consumption alone (referred to as the "consumption model"). It is useful to view this model as a special case of the general framework described in Section 3.2, and apply the results developed for that framework to this particular case. This displays the flexibility of the reduced-form model, and provides an alternative approach to some of the duality results obtained exclusively for the consumption model by Peleg and Ryder (1972, 1974). We present this material in Section 3.7. Of particular interest is the result that, in the consumption model, the competitive condition can be split up into two conditions, one involving purely consumption decisions and the other involving purely production decisions.

Weitzman (1976) showed (in a continuous time optimal growth model) that along an optimal program, the net national product at each instant of time represents the annuity equivalent of its dynamic social welfare from that time onwards. In Section 3.8, as an application of the results on price characterization of optimality in the consumption model, we revisit his interesting economic interpretation of the Bellman equation of dynamic programing. We provide a discrete time analog of his result, displaying the elementary nature of the argument needed to obtain it.

As already indicated in the opening paragraph of this section, the scope of our survey is deliberately limited. To help the reader see some of the connections with the literature that we do not cover, Section 3.9 contains some bibliographic comments on the various sections.

3.2 A General Intertemporal Allocation Model

The framework is described by a triplet (Ω, u, δ), where Ω, a subset of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, is a *transition possibility set*, $u : \Omega \to \mathbb{R}$ is a *utility function* defined on this set, and δ is the *discount factor* satisfying $0 < \delta < 1$. A typical element of Ω is written as an ordered pair (x, y); this means that if the current state is x, then it is possible to be in the state y in one period.

We will be using the following assumptions:

(A.1) (i) $(0,0) \in \Omega$; (ii) $(0,y) \in \Omega$ *implies* $y = 0$.

(A.2) Ω *is* (i) *closed, and* (ii) *convex.*

(A.3) *There is* ξ *such that* "$(x,y) \in \Omega$ *and* $\|x\| > \xi$" *implies* "$\|y\| < \|x\|$".

(A.4) *If* $(x,y) \in \Omega$ *and* $x' \geq x$, $0 \leq y' \leq y$, then (i) $(x',y') \in \Omega$ *and* (ii) $u(x',y') \geq u(x,y)$.

(A.5) u *is* (i) *upper semicontinuous and* (ii) *concave* on Ω.

(A.6) *There is* $\zeta \in \mathbb{R}$, *such that* $(x, y) \in \Omega$ *implies* $u(x, y) \geq \zeta$.

A *program* from $y \in \mathbb{R}^n_+$ is a sequence $\{y(t)\}_0^\infty$ such that $y(0) = y$, and $(y(t),\, y(t+1)) \in \Omega$ for $t \geq 0$.

A program $\{y(t)\}_0^\infty$ from $y \in \mathbb{R}^n_+$ is an *optimal program* if

$$\sum_{t=0}^{\infty} \delta^t u(y'(t), y'(t+1)) \leq \sum_{t=0}^{\infty} \delta^t u(y(t), y(t+1))$$

for every program $\{y'(t)\}_0^\infty$ from y.

The following "boundedness properties" of our model are well-known.

(R.1) *Under assumptions* (A.3) *and* (A.4) (i),

(i) *If* $(x, y) \in \Omega$, *then* $\|y\| \leq \max[\xi,\, \|x\|]$.

(ii) *If* $\{y(t)\}_0^\infty$ *is a program from* $y \in \mathbb{R}^n_+$, *then* $\|y(t)\| \leq \max[\xi,\, \|y\|]$ *for* $t \geq 0$.

The existence of an optimal program in this framework is also a standard result.

(R.2) *Under assumptions* (A.1), (A.2), (A.3), (A.4) (i), (A.5) (i) *and* (A.6), *if* $y \in \mathbb{R}^n_+$, *there exists an optimal program from* y.

Given (R.2), there is an optimal program $\{y^*(t)\}_0^\infty$ from each $y \in \mathbb{R}^n_+$. We define

$$V(y) = \sum_{t=0}^{\infty} \delta^t u(y^*(t),\, y^*(t+1))$$

V is known as the *value function.* By (A.4), V is non-decreasing, and by (A.2) and (A.5), V is concave.

3.3 Characterization of Optimal Programs in Terms of Dual Variables

The principal results on duality theory in infinite horizon optimization models relate optimal programs with programs which are "supported" by dual variables known as shadow prices. At the given shadow prices, the "supported" program maximizes the generalized profit at each date among all feasible activities (pairs (x, y) in the transition possibility set) and is called a competitive program. These results are analogous to the first and second fundamental theorems of welfare economics in general equilibrium theory.

The infinite horizon entails that an additional condition, known as the transversality condition, is involved in relating competitive to optimal programs.

Results which provide sufficient conditions (in terms of shadow prices) for a program to be optimal are often useful in checking that a candidate program is optimal, if one has a good idea of shadow prices that support such a program. Such "price characterization" results make duality theory a principal alternative to dynamic programming methods in solving for an optimal program. Results which provide necessary conditions (in terms of shadow prices) for a program to be optimal are useful in inferring qualitative properties of an optimal program without necessarily solving for an optimal program. Together, these results can be a powerful tool in the hands of an optimal growth theorist to address a variety of problems.

This section provides these "price characterization" results in the context of the general intertemporal allocation model described in the previous section.

A sequence $\{y(t),\, p(t)\}_0^\infty$ is a *competitive program* from $y \in \mathbb{R}_+^n$ if $\{y(t)\}_0^\infty$ is a program from y, $p(t) \in \mathbb{R}_+^n$ for $t \geq 0$, and for all $t \geq 0$ we have

$$\begin{aligned} &\delta^t u(y(t),\, y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq\ &\delta^t u(x, y) + p(t+1)y - p(t)x \quad \text{for all } (x, y) \in \Omega \end{aligned} \tag{3.1}$$

A competitive program $\{y(t),\, p(t)\}_0^\infty$ from $y \in \mathbb{R}_+^n$ is said to satisfy the *transversality condition* if

$$\lim_{t\to\infty} p(t)\, y(t) = 0 \tag{3.2}$$

3.3.1 When Are Competitive Programs Optimal?

Theorem 3.3.1. *If $\{y(t),\, p(t)\}_0^\infty$ is a competitive program from $y \in \mathbb{R}_+^n$ which satisfies the transversality condition, then $\{y(t)\}_0^\infty$ is an optimal program from y.*

Proof. Let $\{y'(t)\}_0^\infty$ be any program from y. Using (3.1), we have for $t \geq 0$:

$$\begin{aligned} &\delta^t[u(y'(t), y'(t+1)) - u(y(t), y(t+1))] \\ &\leq [p(t+1)y(t+1) - p(t)y(t)] - [p(t+1)y'(t+1) - p(t)y'(t)] \end{aligned} \tag{3.3}$$

Summing (3.3) from $t = 0$ to $t = T$:

$$\begin{aligned} &\sum_{t=0}^{T} \delta^t[u(y'(t), y'(t+1)) - u(y(t), y(t+1))] \\ &\leq [p(T+1)y(T+1) - p(0)y(0)] - [p(T+1)y'(T+1) - p(0)y'(0)] \\ &= p(T+1)y(T+1) - p(T+1)y'(T+1) \\ &\leq p(T+1)y(T+1) \end{aligned} \tag{3.4}$$

Since the quantity $\sum_{t=0}^{T} \delta^t u(y'(t), y'(t+1))$ converges as $T \to \infty$, and so does $\sum_{t=0}^{T} \delta^t u(y(t), y(t+1))$, we can take limits on both sides of (3.4), by using (3.2), and we have:

$$\sum_{t=0}^{\infty} \delta^t u(y'(t), y'(t+1)) - \sum_{t=0}^{\infty} \delta^t u(y(t), y(t+1)) \leq 0$$

which proves that $\{y(t)\}_0^\infty$ is an optimal program from y.

Remarks:

(i) It is worth noting that the above result does not depend on the convexity of the transition possibility set or the concavity of the utility function.

(ii) In Theorem 3.3.1, the transversality condition (3.2) can be replaced by

$$\liminf_{t\to\infty} p(t)y(t) = 0. \tag{3.5}$$

(iii) The significance of the transversality condition (3.2) was first noted by Malinvaud (1953) in his study of intertemporal *efficiency*. Since then, it has been extensively used in the study of intertemporal optimality (as well as efficiency). The extremely simple method of proof is a variant of Malinvaud's proof in the study of efficiency; it was effectively introduced in the multisectoral optimality literature most notably by Gale (1967).

(iv) Notice that the convergence of the discounted utility sum is not essential to the *method*. Thus, in general, we can define a program $\{y(t)\}_0^\infty$ from y to be optimal (in Brock's (1970) terminology "weakly-maximal") if

$$\liminf_{T\to\infty} \sum_{t=0}^{T} \delta^t [u(y'(t), y'(t+1)) - u(y(t), y(t+1))] \leq 0$$

for every program $\{y'(t)\}_0^\infty$ from y. Then (3.1) and (3.5) lead to the optimality of $\{y(t)\}_0^\infty$ by the same method. The point to be noted is that, in this general form, *no assumptions* are needed on Ω, u, δ.

3.3.2 When Are Optimal Programs Competitive?

The converse to Theorem 3.3.1 relies heavily on the "convex structure" of the model. The proof we report here follows closely the approach of Weitzman (1973): the interesting features of his technique of proof are (a) the use of an induction argument to obtain the "dual variables", and (b) the combination of the dynamic programming approach exploiting the value function, with the duality approach exploiting the separation theorem. These ideas are formalized in Lemmas 1 and 2 below, which are then used to obtain Theorem 3.3.2, the basic result of this subsection.

Lemma 3.3.1. *Suppose $\{\bar{y}(t)\}_0^\infty$ is an optimal program from $\bar{y} >> 0$. Then there is $p(0) \in \mathbb{R}_+^n$ such that:*

$$V(\bar{y}(0)) - p(0)\bar{y}(0) \geq V(y) - p(0)y \quad \text{for all } y \in \mathbb{R}_+^n. \tag{3.6}$$

Proof. Define the sets A and B as follows.

$A = \{(\alpha, \beta) \in \mathbb{R}^{n+1} : V(y) - V(\bar{y}(0)) \geq \alpha,\ (\bar{y}(0) - y) \geq \beta$ for some $y \in \mathbb{R}_+^n\}$

$B = \{(\alpha, \beta) \in \mathbb{R}^{n+1} : (\alpha, \beta) >> 0\}$

Clearly, A and B are non-empty and convex. Also $A \cap B = \phi$. For if there is $(\alpha, \beta) \in A \cap B$, then there is $y \in \mathbb{R}_+^n$, such that $V(y) > V(\bar{y}(0))$

and $\bar{y}(0) >> y$. By the "free-disposal" assumption (A.4), $\bar{y}(0) >> y$ implies $V(\bar{y}(0)) \geq V(y)$, which contradicts $V(y) > V(\bar{y}(0))$.

By a standard separation theorem (see, for example, Theorem 3.5, p. 35 in Nikaido (1968)) there is $(\mu, \nu) \in \mathbb{R}_+^{n+1}$, with $(\mu, \nu) \neq 0$, such that:

$$\mu\alpha + \nu\beta \leq 0 \qquad \text{for all } (\alpha, \beta) \in A \tag{3.7}$$

Thus, using the definition of A, we have:

$$\mu(V(y) - V(\bar{y}(0)) + \nu(\bar{y}(0) - y) \leq 0 \text{ for all } y \in \mathbb{R}_+^n \tag{3.8}$$

We claim that $\mu \neq 0$. For if $\mu = 0$, then $\nu > 0$ and using (3.8),

$$\nu(\bar{y}(0) - y) \leq 0 \text{ for all } y \in \mathbb{R}_+^n \tag{3.9}$$

But since $\bar{y}(0) = y >> 0$, we can pick $y = \bar{y}(0)/2$ to contradict (3.9). This establishes our claim, so that $\mu > 0$.

Define $p(0) = (\nu/\mu)$, and use (3.8) to get

$$[V(y) - V(\bar{y}(0))] + p(0)(\bar{y}(0) - y) \leq 0 \text{ for all } y \in \mathbb{R}_+^n$$

which, after transposition of terms, is (3.6).

We call $\hat{x} \in \mathbb{R}_+^n$ *sufficient* if there is $\hat{y} \in \mathbb{R}_{++}^n$, such that $(\hat{x}, \hat{y}) \in \Omega$.

Lemma 3.3.2. *Suppose $\{\bar{y}(t)\}_0^\infty$ is an optimal program from $\bar{y}$. Suppose, also, that there is some sufficient vector $\hat{x}$ in $\mathbb{R}_+^n$. If there is some $t \geq 0$, and $p(t) \in \mathbb{R}_+^n$ such that:*

$$\delta^t V(\bar{y}(t)) - p(t)\bar{y}(t) \geq \delta^t V(y) - p(t)y \quad \text{for all } \; y \in \mathbb{R}_+^n \tag{3.10}$$

then there is $p(t+1) \in \mathbb{R}_+^n$ such that:

$$\delta^{t+1} V(\bar{y}(t+1)) - p(t+1)\bar{y}(t+1) \geq \delta^{t+1} V(y) - p(t+1)y \quad \text{for all } y \in \mathbb{R}_+^n \tag{3.11}$$

and:

$$\begin{aligned} &\delta^t u(\bar{y}(t), \bar{y}(t+1)) + p(t+1)\bar{y}(t+1) - p(t)\bar{y}(t) \\ &\geq \delta^t u(x, y) + p(t+1)y - p(t)x \quad \text{for all } (x, y) \in \Omega \end{aligned} \tag{3.12}$$

Proof. Since $\{\bar{y}(t)\}_0^\infty$ is an optimal program, we have $V(\bar{y}(t)) = u(\bar{y}(t), \bar{y}(t+1)) + \delta V(\bar{y}(t+1))$. Also, for all $(x, y) \in \Omega$, we have $V(x) \geq u(x, y) + \delta V(y)$. Using these facts in (3.10), we have

$$\begin{aligned} \theta(t+1) &\equiv \delta^t u(\bar{y}(t), \bar{y}(t+1)) + \delta^{t+1} V(\bar{y}(t+1)) - p(t)\bar{y}(t) \\ &\geq \delta^t u(x, y) + \delta^{t+1} V(y) - p(t)x \quad \text{for all } \; (x, y) \in \Omega \end{aligned}$$

Thus,

$$\theta(t+1) - \delta^t u(x, y) + p(t)x \geq \delta^{t+1} V(y) \quad \text{for all } \; (x, y) \in \Omega \tag{3.13}$$

Define two sets A and B as follows:

$A = \{(\alpha, \beta) \in \mathbb{R}^{n+1} : \alpha \leq \delta^{t+1}V(y') - [\theta_{t+1} - \delta^t u(x,y) + p(t)x]$ and $\beta \leq (y - y')$, for some $(x,y) \in \Omega$, and for some $y' \in \mathbb{R}^n_+\}$

$B = \{(\alpha, \beta) \in \mathbb{R}^{n+1} : (\alpha, \beta) >> 0\}$

Clearly, A and B are non-empty and convex (since u is concave on Ω and V is concave on $\mathbb{R}^n_+$). Also, since V is non-decreasing, we can use (3.13) to infer that $A \cap B = \phi$. Hence, by a standard separation theorem (see, for example, p. 35 in Nikaido (1968)), there is $(\mu, \nu) \in \mathbb{R}^{n+1}_+$, with $(\mu, \nu) \neq 0$, such that:

$$\mu\alpha + \nu\beta \leq 0 \quad \text{for all } (\alpha, \beta) \in A \tag{3.14}$$

Using the definition of A and (3.14), we have

$$\begin{aligned} \mu[\theta_{t+1} - \delta^t u(x,y) + p(t)x] - \nu y \quad &\geq \quad \mu\delta^{t+1}V(y') - \nu y' \\ \text{for all } (x,y) \in \Omega \text{ and all } y' \in \mathbb{R}^n_+. \end{aligned} \tag{3.15}$$

Put $x = \bar{y}(t)$ and $y = \bar{y}(t+1)$ in (3.15) to get:

$$\mu[\delta^{t+1}V(\bar{y}(t+1))] - \nu\bar{y}(t+1) \geq \mu[\delta^{t+1}V(y')] - \nu y' \quad \text{for all } y' \in \mathbb{R}^n_+ \tag{3.16}$$

Put $y' = \bar{y}(t+1)$ in (3.15) to get:

$$\begin{aligned} &\mu[\delta^t u(\bar{y}(t), \bar{y}(t+1)) - p(t)\bar{y}(t)] + \nu\bar{y}(t+1) \\ \geq \quad &\mu[\delta^t u(x,y) - p(t)x] + \nu y \quad \text{for all } (x,y) \in \Omega \end{aligned} \tag{3.17}$$

We claim now that $\mu \neq 0$. For if $\mu = 0$, then by (3.16), $\nu\bar{y}(t+1) \leq \nu y$ for all $y' \in \mathbb{R}^n_+$, and by (3.17), $\nu\bar{y}(t+1) \geq \nu y$ for all y, such that $(x,y) \in \Omega$ for some x. Thus,

$$\nu\bar{y}(t+1) = \nu y \quad \text{for all } y \text{ such that } (x,y) \in \Omega \text{ for some } x \tag{3.18}$$

Since there is a sufficient vector $\hat{x}$, there is $\hat{y} >> 0$, such that $(\hat{x}, \hat{y}) \in \Omega$. Also, $(\hat{x}, 0) \in \Omega$ by "free-disposal" in Ω. Using these facts in (3.18), we have $0 = \nu 0 = \nu\bar{y}(t+1) = \nu\hat{y}$, so that $\nu = 0$. Thus, we get $(\mu, \nu) = 0$, a contradiction. This establishes our claim, and we have $\mu > 0$.

Defining $p(t+1) = (\nu/\mu)$, we can use (3.16) to get (3.11), and (3.17) to get (3.12), establishing the Lemma.

Theorem 3.3.2. *Suppose $\{\bar{y}(t)\}_0^\infty$ is an optimal program from $\bar{y} >> 0$. Suppose, also, that there is some sufficient vector $\hat{x}$ in $\mathbb{R}^n_+$. Then, there is a sequence $\{p(t)\}_0^\infty$ with $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, such that:*

$$\delta^t V(\bar{y}(t)) - p(t)\bar{y}(t) \geq \delta^t V(y) - p(t)y \quad \text{for all } y \in \mathbb{R}^n_+ \tag{3.19}$$

and:

$$\begin{aligned} &\delta^t u(\bar{y}(t), \bar{y}(t+1)) + p(t+1)\bar{y}(t+1) - p(t)\bar{y}(t) \\ &\geq \delta^t u(x,y) + p(t+1)y - p(t)x \quad \text{for all } (x,y) \in \Omega \end{aligned} \tag{3.20}$$

and:

$$\lim_{t \to \infty} p(t)\bar{y}(t) = 0 \tag{3.21}$$

Proof. Using Lemmas 1 and 2, there is a sequence $\{p(t)\}_0^\infty$ with $p(t) \in \mathbb{R}_+^n$ for $t \geq 0$,such that (3.19) and (3.20) hold. Putting $y = 0$ in (3.19), we have

$$\delta^t[V(\bar{y}(t)) - V(0)] \geq p(t)\bar{y}(t) \quad \text{for} \quad t \geq 0 \tag{3.22}$$

Denoting $\max[\xi, \|\bar{y}\|]$ by $B(\bar{y})$ and defining $z = B(\bar{y})e$, where $e = (1, ..., 1)$ in $\mathbb{R}_+^n$, we have $\bar{y}(t) \leq z$ for $t \geq 0$, and so $V(\bar{y}(t)) \leq V(z)$ for $t \geq 0$. Using this in (3.22),

$$\delta^t[V(z) - V(0)] \geq p(t)\bar{y}(t) \geq 0 \quad \text{for} \quad t \geq 0.$$

Now, since $\delta^t \to 0$ as $t \to \infty$, we have $p(t)\bar{y}(t) \to 0$ as $t \to \infty$, which establishes (3.21).

Remarks:

(i) Peleg (1970) establishes a version of Theorem 3.3.2 by applying the separation theorem in the space of all bounded infinite sequences (of vectors in $\mathbb{R}^n$), known as ℓ_∞^n. This method is also followed in Peleg and Ryder (1972).

(ii) In the statement of Theorem 3.3.2, the initial stock, $\bar{y}$, is assumed to be strictly positive, and it is also assumed that there is some sufficient vector, $\hat{x}$ in $\mathbb{R}_+^n$. Under these assumptions, we note that we can find $0 < \lambda < 1$, such that $\lambda\hat{x} \leq \bar{y}$. Now, since there is $\hat{y} >> 0$, such that $(\hat{x}, \hat{y}) \in \Omega$, we have $(\lambda\hat{x}, \lambda\hat{y}) \in \Omega$, and by free-disposal, $(\bar{y}, \lambda\hat{y}) \in \Omega$. Since $\lambda\hat{y} >> 0$, we see that $\bar{y}$ itself is a sufficient vector. Thus, under the assumptions of Theorem 3.3.2, we have (a) $\bar{y} >> 0$ and (b) $\bar{y}$ is a sufficient vector. On the other hand, if $\bar{y} >> 0$ and $\bar{y}$ is a sufficient vector, then the assumptions used in Theorem 3.3.2 are obvously satisfied.

(iii) Conditions (3.19) and (3.21) in the above result are not "independent". For a competitive program it can be shown that (3.19) is equivalent to (3.21). That (3.19) implies (3.21) is clear from the proof of Theorem 3.3.2. The converse implication can be derived by following the proof of Theorem 3.3.1.

3.3.3 An Example

Theorem 3.3.1 shows that a competitive program satisfying a transversality condition is optimal, and Theorem 3.3.2 shows that an optimal program is competitive and satisfies a transversality condition. This still does not settle the question of whether the transversality condition is needed in the statement of Theorem 3.3.1 to make it valid. It is logically possible, for example, that a competitive program automatically satisfies the transversality condition and is therefore optimal. (For more on this line of thought, see Section 3.6 below). In this subsection, to settle this issue, we give an example of a framework (which is a special case of the one described in Section 3.2) and a competitive program in that framework (with a uniquely defined associated price sequence) which violates the transversality condition and is not optimal. Thus, in general, Theorem 3.3.1 would be invalid if the transversality condition is dropped from its statement.

The framework is the standard aggregative neoclassical growth model, which is described by (f, w, δ), where f is the production function, satisfying:

$$f(x) = 4x^{1/2} \text{ for all } x \geq 0$$

w is the welfare function satisfying:

$$w(c) = 2c^{1/2} \text{ for all } c \geq 0$$

and δ is the discount factor, satisfying $\delta = 1/2$. To convert this model to the framework analyzed in Section 3.2, we can define the transition possibility set by $\Omega = \{(x,y) \in \mathbb{R}^2_+ : y \leq f(x)\}$, and the utility function by $u(x,y) = w(x - f^{-1}(y))$ for all $(x,y) \in \Omega$. [For more on this kind of conversion, see Section 3.7 below].

Define a sequence $\{k(t)\}$ as follows: $k(0) = 2, k(1) = 4\sqrt{2} - 1$, and for $t \geq 0$,

$$k(t+2) = f(k(t+1)) - \frac{[f(k(t)) - k(t+1)]}{k(t+1)} \tag{3.23}$$

We first verify that (3.23) does, in fact, uniquely define a sequence. To this end, we claim that, if for some $t \geq 0$, we have $(k(t), k(t+1))$ satisfying $k(t) > 1$ and $f(k(t)) > k(t+1) > k(t)$, then $k(t+2)$, defined uniquely by (3.23) satisfies:

$$k(t+1) > 1 \text{ and } f(k(t+1)) > k(t+2) > k(t+1) \tag{3.24}$$

First, we have $k(t+1) > k(t) > 1$ by hypothesis, so $k(t+1) > 1$. Second, we have:

$$0 < \frac{[f(k(t)) - k(t+1)]}{k(t+1)} < [f(k(t)) - k(t+1)]$$

so that (3.23) implies that (i) $k(t+2) > f(k(t+1)) - [f(k(t)) - k(t+1)] > k(t+1)$, since $k(t+1) > k(t)$ and f is increasing; and (ii) $k(t+2) < f(k(t+1))$. This establishes our claim.

Since $(k(0), k(1)) = (2, 4\sqrt{2} - 1)$ satisfies $k(0) > 1$ and $f(k(0)) = 4\sqrt{2} > 4\sqrt{2} - 1 = k(1) > 2 = k(0)$, we can use (3.24) repeatedly to uniquely define $\{k(t)\}$ by (3.23). Further, for all $t \geq 0$, we must have (3.24) holding along such a sequence.

Note that $\{k(t\}$ is monotonically increasing and bounded above by $\xi = 16$, so it must converge to some $k > 1$. Then, using (3.23) and (3.24), we must have $k = f(k) - \{[f(k) - k]/k\}$ and $f(k) \geq k$ respectively, so that $f(k) - k = \{[f(k) - k]/k\}$, and consequently $f(k) = k$ (since $k \neq 1$). Thus $k = \xi = 16$.

Define $\{x(t), y(t), c(t)\}$ as follows: $x(0) = k(0) = 2, y(0) = 4, c(0) = 2$, and for $t \geq 1$,

$$x(t) = k(t), y(t) = f(x(t-1)), c(t) = y(t) - x(t) \tag{3.25}$$

Note that, by (3.24), we have $c(t) > 0$ for $t \geq 0$, and by (3.23), we have:

$$w'(c(t)) = \delta f'(x(t)) w'(c(t+1)) \text{ for all } t \geq 0 \tag{3.26}$$

the Ramsey-Euler equations for this framework. It is easy to check now that $\{y(t)\}$ is a program from $y(0) = 4$, and that (using the concavity of f and w, and (3.26)), $\{y(t), p(t)\}$ is a competitive program at the uniquely defined price sequence $\{p(t)\}$ given by:

$$p(t) = \delta^t w'(c(t)) \quad \text{for } t \geq 0 \tag{3.27}$$

By (3.23) and $k(t) > 1$ for $t \geq 0$, we have $c(t+1) < c(t)$ for $t \geq 1$, and clearly $c(1) = 1 < c(0) = 2$. Thus, $c(t) \leq 2$ for $t \geq 0$, so that $u(y(t), y(t+1)) \leq w(2) = 2\sqrt{2}$ for all $t \geq 0$. However, the sequence $\{y'(t)\}$ defined by $y'(t) = 4$ for all $t \geq 0$ is clearly a program from $y'(0) = 4$, with $u(y'(t), y'(t+1)) = w(3) = 2\sqrt{3}$ for all $t \geq 0$. Thus, $\{y(t)\}$ is not an optimal program from $y(0) = 4$.

Since $x(t) = k(t) \to 16$ as $t \to \infty$, there is T such that $k(t) \geq 3$, and so $f'(x(t)) \leq (2/3)$ for all $t \geq T$. Using (3.26) and (3.27), we see that:

$$p(t+1) = p(0) / \prod_{s=0}^{t} f'(x(s))$$

and so $p(t) \to \infty$ as $t \to \infty$. Since $y(t) = f(x(t-1)) \to 16$ as $t \to \infty$, we have $p(t)y(t) \to \infty$ as $t \to \infty$, a violation of the transversality condition.

3.4 Duality Theory for Stationary Optimal Programs

3.4.1 Existence of a Stationary Optimal Stock via Duality Theory

The question of existence of a non-trivial stationary optimal stock has been discussed extensively in the literature. Two treatments of the subject can be found in Sutherland (1970) and Khan and Mitra (1986), who use a purely primal approach, and Flynn (1980) and McKenzie (1982), who use the dual variable approach. As an illustration of the power of duality methods, we will provide an exposition of the topic using the latter approach.

An optimal program $\{y(t)\}_0^\infty$ from $y \in \mathbb{R}_+^n$ is a *stationary optimal program* if $y(t) = y(t+1)$ for $t \geq 0$. A *stationary optimal stock* is an element $y \in \mathbb{R}_+^n$, such that $\{y\}_0^\infty$ is a stationary optimal program. It is *non-trivial* if $u(y, y) > u(0, 0)$.

A *discounted golden-rule stock* is a stock $\hat{y}$ with $(\hat{y}, \hat{y}) \in \Omega$, such that:

$$u(\hat{y}, \hat{y}) \geq u(x, y) \quad \text{for all } (x, y) \in \phi(\hat{y})$$

where $\phi(\hat{y}) = \{(x, y) \in \Omega : \delta y - x \geq \delta\hat{y} - \hat{y}\}$.

A *modified golden-rule* is a pair $(\hat{y}, \hat{p})$ with $(\hat{y}, \hat{y}) \in \Omega$, $\hat{p} \in \mathbb{R}_+^n$ such that for all $(x, y) \in \Omega$,

$$u(\hat{y}, \hat{y}) + \delta\hat{p}\hat{y} - \hat{p}\hat{y} \geq u(x, y) + \delta\hat{p}y - \hat{p}x$$

An economy is $\delta - productive$ if there exists $(a, b) \in \Omega$ such that $\delta b >> a$. It is $\delta u - productive$ if there exists $(a, b) \in \Omega$ such that $\delta b >> a$ and $u(\delta b, b) > u(0, 0)$.

It can be shown, by using the Kakutani fixed point theorem, that there exists a discounted golden rule stock (see Theorem 3.4.1 below). Further, when the economy is $\delta - productive$, any discounted golden rule stock, $\hat{y}$, can be supported by a price vector, $\hat{p}$, so that the pair $(\hat{y}, \hat{p})$ is a modified golden-rule, and $\{\hat{y}\}$ is a stationary optimal program from $\hat{y}$ (Theorem 3.4.2). Finally, when the economy is $\delta u - productive$, then a simple consequence of Theorem 3.4.2 is that there exists a non-trivial stationary optimal stock (Corollary 3.4.1).

Theorem 3.4.1. *There exists a discounted golden-rule stock.*

The proof of Theorem 3.4.1 can be obtained from Khan and Mitra (1986), and with some modifications, from McKenzie (1982).

Theorem 3.4.2. *(i) If the economy is $\delta - productive$, and if $\hat{y}$ is a discounted golden-rule stock, then there is $\hat{p} \in \mathbb{R}^n_+$ such that $(\hat{y}, \hat{p})$ is a modified golden-rule.*

(ii) if $(\hat{y}, \hat{p})$ is a modified golden-rule, then $\{\hat{y}\}$ is a stationary optimal program from $\hat{y}$.

Proof. (i) Define the sets A and B as follows:

$$\begin{aligned} A &= \{(\alpha, \beta) \in \mathbb{R}^{n+1} : \alpha \leq u(x, y) - u(\hat{y}, \hat{y}), \text{ and} \\ \beta &\leq (\delta y - x) - (\delta \hat{y} - \hat{y}) \text{ for some } (x, y) \in \Omega\} \end{aligned}$$

$$B = \{(\alpha, \beta) \in \mathbb{R}^{n+1} : (\alpha, \beta) >> 0\}$$

Note that A and B are non-empty, convex sets in $\mathbb{R}^{n+1}$, and they are disjoint, since $\hat{y}$ is a discounted golden-rule stock. Thus, by a standard separation theorem, there is $(\mu, \nu) \in \mathbb{R}^{n+1}_+$ with $(\mu, \nu) \neq 0$, such that:

$$\mu\alpha + \nu\beta \leq 0 \text{ for all } (\alpha, \beta) \in A$$

This implies that for all $(x, y) \in \Omega$,

$$\mu u(x, y) + \nu(\delta y - x) \leq \mu u(\hat{y}, \hat{y}) + \nu(\delta \hat{y} - \hat{y}) \tag{3.28}$$

We claim that $\mu \neq 0$. For if $\mu = 0$, then $\nu \neq 0$, and (3.28) implies that:

$$\nu(\delta y - x) \leq \nu(\delta \hat{y} - \hat{y}) \text{ for all } (x, y) \in \Omega \tag{3.29}$$

Since the economy is $\delta - productive$, there is $(a, b) \in \Omega$ satisfying $\delta b >> a$. Using this in (3.29), we get $0 < \nu(\delta b - a) \leq \nu(\delta \hat{y} - \hat{y}) \leq 0$, a contradiction. Thus, $\mu > 0$, and defining $\hat{p} = (\nu/\mu)$, we see from (3.28) that $(\hat{y}, \hat{p})$ is a modified golden-rule.

(ii) Define a sequence $\{p(t)\}$ by:

$$p(t) = \delta^t \hat{p} \quad \text{for } t \geq 0$$

Then, using the definition of a modified golden-rule it is easy to check that for all $t \geq 0$, we have:

$$\begin{aligned} & \delta^t u(x,y) + p(t+1)y - p(t)x \\ \leq \quad & \delta^t u(\hat{y},\hat{y}) + p(t+1)\hat{y} - p(t)\hat{y} \quad \text{ for all } (x,y) \in \Omega \end{aligned} \tag{3.30}$$

Further, since $\delta \in (0,1)$, we have:

$$\lim_{t \to \infty} p(t)\hat{y} = 0 \tag{3.31}$$

Thus, by Theorem 3.3.1, $\{\hat{y}\}$ is a stationary optimal program from $\hat{y}$.

We now note the basic result on the existence of a non-trivial stationary optimal stock as a simple consequence of the above results.

Corollary 3.4.1. *If the economy is $\delta u - productive$, then there exists a non-trivial stationary optimal stock.*

Proof. Using Theorem 3.4.1, there is a discounted golden-rule stock, $\hat{y}$. Since the economy is $\delta - productive$, Theorem 3.4.2 can be applied to infer that $\hat{y}$ is a stationary optimal stock. Finally, since the economy is $\delta u - productive$, we can infer that $\hat{y}$ is a non-trivial stationary optimal stock, by definition of a discounted golden rule.

3.4.2 Quasi-Stationary Price Support for Stationary Optimal Programs

Using Theorems 3.4.1 and 3.4.2, we see that there always exists a stationary optimal program $\{\hat{y}\}$, which is supported (in the sense of (3.30)) by a *quasi-stationary* price sequence; that is, by a price sequence of the form $p(t) = \delta^t \hat{p}$ for $t \geq 0$. It turns out that any stationary optimal program $\{\hat{y}\}$ can be supported by a quasi-stationary price sequence, provided $(\hat{y}, \hat{y})$ is in the interior of Ω. That is, compared to Theorem 3.3.2, one can choose the supporting price sequence from a more restricted set when the optimal program happens to be stationary. Our exposition of this result follows Sutherland (1967) and McKenzie (1986).

Theorem 3.4.3. *Suppose $\{\hat{y}\}$ is a stationary optimal program from $\hat{y}$, and $(\hat{y}, \hat{y}) \in int\Omega$. Then, there is $\hat{p}$ such that:*

(i) $(\hat{y}, \hat{p})$ is a modified golden-rule, and

(ii) defining $\hat{p}(t) = \delta^t \hat{p}$ for $t \geq 0, \{\hat{y}, \hat{p}(t)\}$ is a competitive program from $\hat{y}$, which satisfies the transversality condition.

Proof. Using Theorem 3.3.2, we know that there is a sequence $\{p(t)\}$, with $p(t) \in \mathbb{R}^n_+$ such that $\{\hat{y}, p(t)\}$ is a competitive program from $\hat{y}$. Then, for each $t \geq 0$, we have:

$$\begin{aligned} & \delta^t u(x,y) + p(t+1)y - p(t)x \\ \le \; & \delta^t u(\hat{y},\hat{y}) + p(t+1)\hat{y} - p(t)\hat{y} \quad \text{for all } (x,y) \in \Omega \end{aligned} \tag{3.32}$$

Denoting $p(t)/\delta^t$ by $q(t)$ for each $t \geq 0$, we have:

$$\begin{aligned} & u(x,y) + \delta q(t+1)y - q(t)x \\ \le \; & u(\hat{y},\hat{y}) + \delta q(t+1)\hat{y} - q(t)\hat{y} \quad \text{for all } (x,y) \in \Omega \end{aligned} \tag{3.33}$$

Since $(\hat{y},\hat{y}) \in int\Omega$, we have $B > 0$, such that $||q(t)|| \le B$ for all $t \geq 0$.

Averaging the first $(T+1)$ inequalities in (3.33) gives:

$$u(x,y) - u(\hat{y},\hat{y}) \leqq \delta Q(T)(\hat{y} - y) - P(T)(\hat{y} - x) \;\; \text{for all } (x,y) \in \Omega \tag{3.34}$$

where

$$P(T) = \frac{1}{T+1}(q(0) + q(1) + ... + q(T))$$

and

$$\begin{aligned} Q(T) &= \frac{1}{T+1}(q(1) + ... + q(T+1)) \\ &= P(T) + \frac{1}{T+1}(q(T+1) - q(0)) \end{aligned}$$

Clearly, $||P(T)|| \le B$ for all $T \geq 0$; so, there is a subsequence $\{T_i\}$, $i = 1, 2, ...$, such that $P(T_i) \to \hat{p} \geqq 0$. Then $Q(T_i)$ also converges to $\hat{p}$, and (3.34) yields:

$$u(x,y) - u(\hat{y},\hat{y}) \leqq \delta \hat{p}(\hat{y} - y) - \hat{p}(\hat{y} - x) \;\; \text{for all } (x,y) \in \Omega \tag{3.35}$$

Thus $(\hat{y},\hat{p})$ is a modified golden-rule, establishing (i). The result in (ii) follows by using the proof of (ii) in Theorem 3.4.2.

Remarks:

(i) Suppose $(\hat{y},\hat{y}) \in int\Omega$; then, $\{\hat{y}\}$ is a stationary optimal program if and only if there is $\hat{p} \in \mathbb{R}^n_+$ such that $(\hat{y},\hat{p})$ is a modified golden-rule. This follows from Theorem 3.4.2 (ii) and Theorem 3.4.3 (i).

(ii) Suppose $(\hat{y},\hat{y}) \in int\Omega$, and $\{\hat{y}\}$ is a stationary optimal program. Then, from Theorem 3.4.3 (i), it follows that $\hat{y}$ is also a discounted golden-rule stock.

3.5 Replacing the Transversality Condition by a Period-by-Period Condition

The transversality condition used in the price characterization results of optimality (Theorems 3.3.1 and 3.3.2 of Section 3.3) is an asymptotic condition. It cannot be verified in a finite number of periods, however large the number of periods might be. It is, therefore, of some interest to investigate whether the transversality condition can be replaced in such characterization results by a condition which might convey some information about optimality in a finite number of periods (where the finite number of periods can be arbitrarily "large"). It turns out that, for the class of stationary models we are considering, there is a convenient period-by-period condition which can replace the transversality condition in the price characterization theorems. Our exposition of this result follows Dasgupta and Mitra (1988).

To describe the results of this section, it is convenient to adopt the following convention. If $\{y(t), p(t)\}$ is a competitive program, then we denote the *current value price sequence* associated with it by $\{q(t)\}$, where $q(t) = p(t)/\delta^t$ for $t \geq 0$. If $\{\hat{y}, \hat{q}\}$ is a modified golden-rule, then the *present value price sequence* associated with it is denoted by $\{\hat{p}(t)\}$, where $\hat{p}(t) = \delta^t \hat{q}$ for $t \geq 0$.

If $\{y(t)\}$ is an optimal program, and $(\hat{y}, \hat{q})$ is a modified golden-rule, then Theorem 3.3.2 can be used to show the existence of a price sequence $\{p(t)\}$ such that $\{y(t), p(t)\}$ is a competitive program, and the following inequality holds:

$$(q(t) - \hat{q})(y(t) - \hat{y}) \leq 0 \quad \text{for all } t \geq 0 \tag{3.36}$$

(see Theorem 3.5.1 below). This raises the following question: if $(\hat{y}, \hat{q})$ is a modified golden-rule, and $\{y(t), p(t)\}$ is a competitive program, such that the period-by-period condition (3.36) is satisfied, then is $\{y(t)\}$ an optimal program ? If the stock $\hat{y}$ is "proportionately expansible", then the answer is in the affirmative (see Theorem 3.5.2 below). The results are useful in identifying *non-optimal* competitive programs. That is, if $\{y(t), p(t)\}$ is a competitive program, which is *not* optimal, then it must violate (3.36) for some period. Further, if $\{y(t), p(t)\}$ is a competitive program, for which $\{p(t)\}$ is the unique associated price sequence, and it violates (3.36) for some period t, then it can be pronounced to be non-optimal. Note that this would not be possible by using Theorem 3.3.2.

Theorem 3.5.1. *Suppose there exists a sufficient vector. Let $\{y(t)\}$ be an optimal program from $\bar{y} >> 0$, and let $(\hat{y}, \hat{q})$ be a modified golden-rule. Then, there is a price sequence $\{p(t)\}$, with $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, such that $\{y(t), p(t)\}$ is a competitive program, and:*

$$(q(t) - \hat{q})(y(t) - \hat{y}) \leq 0 \quad \text{for all } t \geq 0$$

Proof. By Theorem 3.3.2, there is a price sequence $\{p(t)\}$, with $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, such that $\{y(t), p(t)\}$ is a competitive program, and:

$$V(y(t)) - q(t)y(t) \geq V(\hat{y}) - q(t)\hat{y} \quad \text{for all } t \geq 0 \tag{3.37}$$

Since $\{\hat{y}, \hat{q}\}$ is a modified golden-rule, $\{\hat{y}, \hat{p}(t)\}$ is a competitive program, and $\hat{p}(t)\hat{y} = \delta^t \hat{q}\hat{y} \to 0$ as $t \to \infty$. Thus, using remark (iii) following Theorem 3.3.2, we have:

$$V(\hat{y}) - \hat{q}\hat{y} \geq V(y(t)) - \hat{q}y(t) \text{ for all } t \geq 0 \tag{3.38}$$

Adding (3.37) and (3.38) and transposing terms yields the desired result.

For the converse result, we first establish some properties that hold for competitive programs (Lemma 3.5.1), and then derive Theorem 3.5.2 from it.

Lemma 3.5.1. *Suppose* $\{y(t), p(t)\}$ *is a competitive program, and* $(\hat{y}, \hat{q})$ *is a modified golden-rule. Then :*

$$(p(t+1) - \hat{p}(t+1))(y(t+1) - \hat{y}) \geq (p(t) - \hat{p}(t))(y(t) - \hat{y}) \quad \text{for all } t \geq 0 \tag{3.39}$$

Further, if (3.36) holds, then:

$$[(p(t+1)-\hat{p}(t+1))(y(t+1)-\hat{y})-(p(t)-\hat{p}(t))(y(t)-\hat{y})] \to 0 \quad \text{as } t \to \infty \tag{3.40}$$

Proof. Since $\{y(t), p(t)\}$ is competitive, we have:

$$\begin{aligned} & \delta^t u(y(t), y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq \; & \delta^t u(\hat{y}, \hat{y}) + p(t+1)\hat{y} - p(t)\hat{y} \quad \text{for all } t \geq 0 \end{aligned} \tag{3.41}$$

Since $\{\hat{y}, \hat{p}(t)\}$ is competitive, we have:

$$\begin{aligned} & \delta^t u(\hat{y}, \hat{y}) + \hat{p}(t+1)\hat{y} - \hat{p}(t)\hat{y} \\ \geq \; & \delta^t u(y(t), y(t+1)) + \hat{p}(t+1)y(t+1) - \hat{p}(t)y(t) \quad \text{for all } t \geq 0 \end{aligned} \tag{3.42}$$

Adding (3.41) and (3.42) and transposing terms yields (3.39).

Denoting $(p(t) - \hat{p}(t))(y(t) - \hat{y})$ by $\mu(t)$ for $t \geq 0$, we see (from (3.39)) that $\{\mu(t)\}$ is a monotonically non-decreasing sequence. If (3.36) holds, this sequence is bounded above by 0. So, $\mu(t)$ converges as $t \to \infty$. Clearly, this implies that (3.40) must hold.

A stock $y \in \mathbb{R}^n_+$ is called *expansible* if there is $y' >> y$, such that $(y, y') \in \Omega$. It is called *proportionately expansible* if there is $\lambda > 1$ such that $(y, \lambda y) \in \Omega$. Clearly, if y is expansible, it is proportionately expansible. Also, note that if $(y, y) \in int\Omega$, then y is expansible.

Theorem 3.5.2. *Suppose* $(\hat{y}, \hat{q})$ *is a modified golden-rule and* $\hat{y}$ *is proportionately expansible. If* $\{y(t), p(t)\}$ *is a competitive program from* $\bar{y}$*, which satisfies (3.36), then:*

(i) $p(t)\hat{y} \to 0$ *as* $t \to \infty$*, and*

(ii) $\{y(t)\}$ *is an optimal program from* $\bar{y}$*.*

Proof. Since $\{y(t), p(t)\}$ is competitive, we have:

$$\begin{aligned} & \delta^t u(y(t), y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq \; & \delta^t u(\hat{y}, \lambda\hat{y}) + p(t+1)\lambda\hat{y} - p(t)\hat{y} \quad \text{for all } t \geq 0 \end{aligned} \tag{3.43}$$

Transposing terms, one gets:

$$\delta^t[u(y(t), y(t+1)) - u(\hat{y}, \lambda\hat{y})] + p(t+1)(y(t+1) - \hat{y}) - p(t)(y(t) - \hat{y})$$

$$\geq p(t+1)(\lambda-1)\hat{y} \tag{3.44}$$

Denoting $(p(t)-\hat{p}(t))(y(t)-\hat{y})$ by $\mu(t)$ for $t \geq 0$, we can write:

$$p(t)(y(t)-\hat{y}) = \hat{p}(t)(y(t)-\hat{y}) + \mu(t) \quad \text{for all } t \geq 0 \tag{3.45}$$

Using (3.45) in (3.44), we obtain:

$$\delta^t[u(y(t),y(t+1)) - u(\hat{y},\lambda\hat{y})] + \hat{p}(t+1)(y(t+1)-\hat{y}) - \hat{p}(t)(y(t)-\hat{y})$$

$$+\mu(t+1) - \mu(t) \geq p(t+1)(\lambda-1)\hat{y} \tag{3.46}$$

Denoting $\max\{\xi, ||\bar{y}||\}$ by B, we have $y(t) \leq Be$, where $e = (1,1,...,1)$ in $\mathbb{R}^n$. Thus, $u(y(t),y(t+1)) \leq u(Be,0)$ for all $t \geq 0$. Then, using Lemma 3.5.1, we note that all the terms on the left hand side of (3.46) converge to zero as $t \to \infty$. This establishes (i), since $\lambda > 1$.

By definition of $\mu(t)$ and (3.36), we have:

$$\begin{aligned} p(t)(y(t)-\hat{y}) &= \hat{p}(t)(y(t)-\hat{y}) + \mu(t) \\ &\leq \hat{p}(t)(y(t)-\hat{y}) \\ &\leq \hat{p}(t)y(t) \end{aligned}$$

Thus, we get:

$$p(t)y(t) \leq p(t)\hat{y} + \hat{p}(t)y(t) \tag{3.47}$$

Since $\|y(t)\| \leq \max[\xi, \|\bar{y}\|]$ *for* $t \geq 0$, and $\delta \in (0,1)$, we have $\hat{p}(t)y(t) \to 0$ as $t \to \infty$. Also, $p(t)\hat{y} \to 0$ as $t \to \infty$ by (i). Thus, by (3.47), we must have $p(t)y(t) \to 0$ as $t \to \infty$. By Theorem 3.3.1, $\{y(t), p(t)\}$ is optimal from $\bar{y}$.

3.6 Are Competitive Programs Optimal?

We have seen that in general an infinite horizon competitive program is not optimal (see the example in Section 3.3), and so Theorem 3.3.1 would be invalid if the transversality condition is dropped from its statement. However, this still leaves open the possibility that for some classes of models, the phenomenon observed in the example does not occur, and all competitive programs are optimal. The approach to identify such models has been to specify a class of transition possibility sets and utility functions such that the (myopic) competitive condition itself restricts the rate at which accumulation of stocks can take place; such a class can be conveniently described by some form of a "reachability" condition. This topic has been investigated by Kurz and Starrett (1970), and Dasgupta and Mitra (1999a,b), among others. We base our discussion here on Dasgupta and Mitra (1999a).

Reachability Condition:

There is an expansible stock $\tilde{y}$ such that, given any competitive program $\{y(t), p(t)\}$, there is a program $\{y^0(t)\}$ from $\tilde{y}$ and a positive integer R such that $y^0_R \geq y_R$.

This condition says that, given a competitive program (from an arbitrary initial stock) it is possible, through pure accumulation if need be, to reach the stocks along the given competitive program at some far enough future date, starting from the expansible stock $\tilde{y}$.

Theorem 3.6.1. *Suppose the Reachability Condition is satisfied. If* $\{y(t), p(t)\}$ *is a competitive program from* $\bar{y} \in \mathbb{R}^n_+$*, then* $\{y(t)\}$ *is an optimal program from* $\bar{y}$.

Proof. Since $\tilde{y}$ is expansible, there is $\tilde{z} >> \tilde{y}$ such that $(\tilde{y}, \tilde{z}) \in \Omega$. Denote $(\tilde{z} - \tilde{y})$ by k; then $k >> 0$. Using the competitive condition, we get, for all $t \geq 0$,

$$\begin{aligned} & \delta^t u(y(t), y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq\ & \delta^t u(\tilde{y}, \tilde{z}) + p(t+1)\tilde{z} - p(t)\tilde{y} \\ =\ & \delta^t u(\tilde{y}, \tilde{z}) + p(t+1)\tilde{y} - p(t)\tilde{y} + p(t+1)k \end{aligned} \tag{3.48}$$

Now consider any $T \geq 2$. From (3.48), we have:

$$\begin{aligned} & \sum_{t=0}^{T-1} \delta^t u(y(t),\ y(t+1)) + p(T)y(T) - p(0)y(0) \\ \geq\ & \sum_{t=0}^{T-1} \delta^t u(\tilde{y}, \tilde{z}) + \sum_{0}^{T-1} p(t+1)k + p(T)\tilde{y} - p(0)\tilde{y} \end{aligned} \tag{3.49}$$

The sequence $\{y''(t), p''(t)\}$ defined by $(y''(t), p''(t)) = (y(T+t), p(T+t))$ for $t \geq 0$ is clearly a competitive program from $y(T)$. By the reachability condition, there is a program $\{y^0(t)\}$ from $\tilde{y}$, and a positive integer R, such that $y^0(R) \geq y''(R) = y(T+R)$. Defining $\{y'(t)\}$ by $y'(t) = \tilde{y}$ for $t = 0, ..., T$, and $y'(t) = y^0(t-T)$ for $t > T$, we see that $\{y'(t)\}$ is a program from $\tilde{y}$, and:

$$y'(T) = \tilde{y} \text{ and } y'(T+R) = y^0(R) \geq y''(R) = y(T+R) \tag{3.50}$$

Applying the competitive condition to $(y'(t),\ y'(t+1)) \in \Omega$ for each $t \geq T$, we have:

$$\begin{aligned} & \delta^t u(y(t), y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq\ & \delta^t u(y'(t), y'(t+1)) + p(t+1)y'(t+1) - p(t)y'(t) \end{aligned} \tag{3.51}$$

From (3.51), we have:

$$\sum_{t=T}^{T+R-1} \delta^t u(y(t), y(t+1)) + p(T+R)y(T+R) - p(T)y(T)$$
$$\geq \sum_{t=T}^{T+R-1} \delta^t u(y'(t), y'(t+1)) + p(T+R)y'(T+R) - p(T)y'(T) \quad (3.52)$$

From (3.49) and (3.52), we have:

$$\sum_{t=0}^{T+R-1} \delta^t u(y(t), x(t+1)) + p(T+R)y(T+R) - p(0)y(0)$$
$$\geq \sum_{t=0}^{T-1} \delta^t u(\tilde{y}, \tilde{z}) + \sum_{t=T}^{T+R-1} \delta^t u(y'(t), y'(t+1)) + \sum_{t=0}^{T-1} p(t+1)k$$
$$+p(T+R)y'(T+R) - p(T)y'(T) + p(T)\tilde{y} - p(0)\tilde{y} \quad (3.53)$$

Since $p(t) \geq 0$, from (3.50) and (3.53), we have:

$$\sum_{t=0}^{T+R-1} \delta^t u(y(t), y(t+1)) - \sum_{t=0}^{T-1} \delta^t u(\tilde{y}, \tilde{z})$$
$$- \sum_{t=T}^{T+R-1} \delta^t u(y'(t), y'(t+1)) + p(0)(\tilde{y} - y(0))$$
$$\geq \sum_{0}^{T-1} p(t+1)k \quad (3.54)$$

Denoting $\max\{\xi, ||\bar{y}||\}$ by B, we have $y(t) \leq Be$, where $e = (1, 1, ..., 1)$ in $\mathbb{R}^n$. Denoting $\max\{\xi, ||\tilde{y}||\}$ by B', we have $y'(t) \leq B'e$, where $e = (1, 1, ..., 1)$ in $\mathbb{R}^n$. Thus, $u(y(t), y(t+1)) \leq u(Be, 0)$, and $u(y'(t), y'(t+1)) \leq u(B'e, 0)$for all $t \geq 0$. Then, using (A.6) and $\delta \in (0, 1)$, we see that the left hand side of (3.54) is uniformly bounded above regardless of the value of T. Since $k >> 0$ and $p(t) \geq 0$ for all t, it follows that $\sum_0^\infty p(t) < \infty$, and so $p(t) \to 0$ as $t \to \infty$. Since $\|y(t)\| \leq \max[\xi, \|\bar{y}\|]$ *for* $t \geq 0$, we have $\lim_{t\to\infty} p(t)y(t) = 0$ and so, by Theorem 3.3.1, the program $\{y(t)\}$ is optimal from $\bar{y}$.

Given the abstract nature of the reachability condition, a simple multi-sectoral model in which it can be directly verified would be helpful. We describe the production side of such a model by an $n \times n$ non-negative matrix $A = (a_{ij})$, where $i = 1, ..., n$ and $j = 1, ..., n$, and a strictly positive vector $b = (b_1, ..., b_n) >> 0$. Here, a_{ij} and b_j are respectively the amounts of the $i-$th good and labor which are required per unit output of the $j-$th good. The total amount of labor available for production is stationary and is normalized to 1. For each $j = 1, ..., n$, it is assumed that there is some $i = 1, ..., n$ such that $a_{ij} > 0$. Thus, each production process requires a positive amount of labor as well as a positive amount of some produced factor. Further, it is assumed that

A is productive; that is, there is some $\tilde{y} >> 0$ such that $\tilde{y} >> A\tilde{y}$ and $b\tilde{y} \leq 1$. This essentially excludes the economically uninteresting case of a production system which is unable to sustain some positive consumption levels for all of the desired goods. The transition possibility set for this economy is:

$$\Omega = \{(x, y) \in \mathbb{R}_+^{2n} : Ay \leq x \text{ and } by \leq 1\}$$

Welfare is derived from consumption, as given by a function $w : \mathbb{R}_+^n \to \mathbb{R}$, which is continuous, concave and monotone on $\mathbb{R}_+^n$. The (reduced form) utility function is then defined by:

$$u(x, y) = w(x - Ay) \quad \text{for all } (x, y) \in \Omega$$

Consider any program $\{y(t)\}$ from $\bar{y} \in \mathbb{R}_+^n$. Denoting $(1/\min_j b_j)$ by B, we have $y(t) \leq Be$ for all $t \geq 1$, where $e = (1, ..., 1) \in \mathbb{R}^n$. Since A is productive, we have $A^t \to 0$ as $t \to \infty$, so we can find a positive integer $R \geq 2$, such that:

$$A^R Be << \tilde{y}$$

Now, define the sequence $\{y'(t)\}$ as follows:

$$\left.\begin{array}{ll} y'(0) = \tilde{y} & \\ y'(t) = A^{R-t} y(R) & \text{for } 1 \leq t \leq R-1 \\ y'(t) = y(t) & \text{for } t \geq R \end{array}\right\}$$

It can be checked (see Dasgupta and Mitra (1999a) for the details) that $\{y'(t)\}$ is a program from $\tilde{y}$. Since $y'(R) = y(R)$, the reachability condition is satisfied.

3.7 Duality Theory in the Consumption Model

A model of optimal growth that has received considerable attention in the literature is one in which utility is derived from consumption alone (referred to as the "consumption model"). In this section we describe the multisectoral version of this model, show how it can be viewed as a special case of the general framework described in Section 3.2, and apply the results developed for that framework to this particular case. In terms of duality theory, the principal difference is that the competitive condition can be split up into two conditions, one involving consumption decisions and the other involving production decisions (see (3.56) and (3.57) below). Our exposition follows Dasgupta and Mitra (1990).

3.7.1 The Model

Consider a framework described by a triplet (Ω, w, δ), where Ω, a subset of $\mathbb{R}_+^n \times \mathbb{R}_+^n$, is the *technology set*, $w : \mathbb{R}_+^n \to \mathbb{R}$ is the period *welfare function*, and

δ is the *discount factor* satisfying $0 < \delta < 1$. A typical element of Ω is written as an ordered pair (x, y), where x represents the inputs of the n goods, and y the outputs producible with inputs x.

We will need the following assumptions:

(B.1) (i) $(0, 0) \in \Omega$; (ii) $(0, y) \in \Omega$ *implies* $y = 0$.

(B.2) Ω *is* (i) *closed, and* (ii) *convex.*

(B.3) *There is* ξ *such that* "$(x, y) \in \Omega$ *and* $\|x\| > \xi$" *implies* "$\|y\| < \|x\|$".

(B.4) *If* $(x, y) \in \Omega$ *and* $x' \geq x$, $0 \leq y' \leq y$, *then* $(x', y') \in \Omega$.

(B.5) w *is* (i) *continuous, and* (ii) *concave.*

(B.6) *If* c, c' *are in* $\mathbb{R}^n_+$, *then* (i) $c' \geq c$ *implies* $w(c') \geq w(c)$, *and* (ii) $c' >> c$ *implies* $w(c') > w(c)$.

A *plan* from $y \in \mathbb{R}^n_+$ is a sequence $\{x(t),\, y(t)\}_0^\infty$ such that

$$y(0) = y;\ 0 \leq x(t) \leq y(t) \text{ and } (x(t), y(t+1)) \in \Omega \text{ for } t \geq 0$$

Associated with a plan $\{x(t), y(t)\}_0^\infty$ from y is a *consumption sequence* $\{c(t)\}_0^\infty$ defined by

$$c(t) = y(t) - x(t) \quad \text{for} \quad t \geq 0$$

A plan $\{\bar{x}(t),\, \bar{y}(t)\}_0^\infty$ from y is an *optimal plan* if

$$\sum_0^\infty \delta^t w(\bar{c}(t)) \geq \sum_0^\infty \delta^t w(c(t)) \tag{3.55}$$

for every plan $\{x(t),\, y(t)\}_0^\infty$ from y.

An optimal plan $\{x(t), y(t)\}_0^\infty$ from y is a *stationary optimal plan* if $(x(t), y(t)) = (x(t+1), y(t+1))$ for $t \geq 0$. In this case we refer to a stationary optimal plan as $\{x, y\}_0^\infty$ with obvious interpretation, and to its associated stationary consumption sequence as $\{c\}_0^\infty$, where $c = y - x$. A *stationary optimal output* is an element $y \in \mathbb{R}^n_+$ such that there is a stationary optimal plan from y. It is *non-trivial* if $w(c) > w(0)$.

A sequence $\{x(t), y(t),\, p(t)\}_0^\infty$ is a *competitive plan* from y if $\{x(t),\, y(t)\}_0^\infty$ is a plan from y, $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, and for $t \geq 0$,

$$\delta^t w(c(t)) - p(t)c(t) \geq \delta^t w(c) - p(t)c \quad \text{for all} \;\; c \in \mathbb{R}^n_+ \tag{3.56}$$

and

$$p(t+1)y(t+1) - p(t)x(t) \geq p(t+1)y - p(t)x \quad \text{for all} \;\; (x, y) \in \Omega \tag{3.57}$$

A *modified golden-rule equilibrium* is a triple $(\hat{x}, \hat{y}, \hat{p})$ with $(\hat{x}, \hat{y}) \in \Omega$, $\hat{p} \in \mathbb{R}^n_+$, such that denoting $(\hat{y} - \hat{x})$ by $\hat{c}$, we have

(i) $\hat{c} \geq 0$

(ii) $w(\hat{c}) - \hat{p}\hat{c} \geq w(c) - \hat{p}c$ for all c in $\mathbb{R}^n_+$

(iii) $\hat{p}(\delta\hat{y} - \hat{x}) \geq \hat{p}(\delta y - x)$ for all $(x, y) \in \Omega$

3.7.2 Conversion to the Format of the General Model

Our objective, in this subsection, is to show that the consumption model can be viewed as a particular case of the general framework of Section 3.2.

To this end, we define a *feasible input correspondence*, $g : \Omega \to \mathbb{R}^n_+$ by

$$g(a, b) = \{x : (x, b) \in \Omega \text{ and } x \leq a\}$$

Note that for each $(a, b) \in \Omega$, $a \in g(a, b)$ so g is non-empty valued. Also, for each $(a, b) \in \Omega$, $g(a, b)$ is a bounded set (by definition) and a closed set, by (B.2).

Next, we define a *utility function*, $u : \Omega \to \mathbb{R}$ by

$$u(a, b) = Max\,\{w(a - x) : x \in g(a, b)\}$$

Note that for each $(a, b) \in \Omega$, $g(a, b)$ is non-empty, compact, and w is continuous. Thus, defining $h(a, b) = \{\bar{x} : \bar{x} \in g(a, b)$, and $w(a - \bar{x}) \geq w(a - x)$ for all $x \in g(a, b)\}$, we note that h is a non-empty valued correspondence on Ω, and $u(a, b)[\equiv w(a - \bar{x})$ for $\bar{x} \in h(a, b)]$ is well-defined on Ω. It can now be shown that, given (B.1) - (B.6), (Ω, u) satisfies (A.1) - (A.6) of Section 3.2 [see Dasgupta and Mitra (1990) for the details].

Next, we want to consider plans in terms of the general framework of Section 3.2. Note that $\{x(t), y(t)\}_0^\infty$ is a plan from y if and ony if $\{y(t)\}_0^\infty$ is a program from y, and $x(t) \in g(y(t), y(t+1))$ for $t \geq 0$. Furthermore, if $\{\bar{x}(t), \bar{y}(t)\}_0^\infty$ is an optimal plan from y, then clearly $\bar{x}(t) \in h(\bar{y}(t),\ \bar{y}(t+1))$ and so $u(\bar{y}(t),$ $\bar{y}(t+1)) = w(\bar{c}(t))$ for $t \geq 0$. Also, if $\{x(t), y(t)\}_0^\infty$ is a plan from y, then $w(c(t)) = w(y(t) - x(t)) \leq u(y(t), y(t+1))$. Using these facts, the inequality in (3.55) can be rewritten as:

$$\sum_0^\infty \delta^t u(\bar{y}(t), \tilde{y}(t+1)) \geq \sum_0^\infty \delta^t u(y(t),\ y(t+1))$$

for every plan $\{x(t),\ y(t)\}_0^\infty$ from y. In other words, $\{\bar{y}(t)\}_0^\infty$ is an optimal program from y. Conversely, if $\{\bar{y}(t)\}_0^\infty$ is an optimal program from y, then defining $\bar{x}(t) \in h(\bar{y}(t),\ \bar{y}(t+1))$ for $t \geq 0$, $\{\bar{x}(t),\ \bar{y}(t)\}_0^\infty$ is clearly an optimal plan from y.

3.7.3 Characterization of Optimal Plans in Terms of Dual Variables

An optimal plan can be characterized as a competitive plan satisfying a transversality condition. The standard references for this result are Peleg (1970) and Peleg and Ryder (1972). The (common) technique of proof of these two papers consists in applying a separation theorem in the space of all bounded infinite sequences (of vectors in $\mathbb{R}^n$). Our main objective in presenting this result is to draw attention to the fact that it can be derived as a special case of Theorems 3.3.1 and 3.3.2, which we have noted for the general model.

We now formally state and prove our characterization results, by using the corresponding results for the general model.

Proposition 3.7.1. *If* $\{x(t),\, y(t),\, p(t)\}_0^\infty$ *is a competitive plan from* y, *and*

$$\lim_{t\to\infty} p(t)\, y(t) = 0 \tag{3.58}$$

then $\{x(t),\, y(t)\}_0^\infty$ *is an optimal plan from* y.

Proof. If $\{x(t),\, y(t),\, p(t)\}_0^\infty$ is a competitive plan from y, then using (3.56), (3.57), and $x(t) = y(t) - c(t)$, one gets:

$$\delta^t w(c(t)) + p(t+1)\, y(t+1) - p(t)y(t)$$

$$\geq \delta^t w(c) + p(t+1)y - p(t)(c+x) \quad \text{for all} \quad (x,y)\ \in, \Omega \text{ and } c \in \mathbb{R}^n_+ \tag{3.59}$$

Note that $x(t) \in g(y(t),\, y(t+1))$, since $(x(t),\, y(t+1)) \in \Omega$, and $x(t) \leq y(t)$. For any $x \in g(y(t),\, y(t+1))$, since $(x(t),\, y(t+1)) \in \Omega$ and $x \leq y(t)$, defining $c = y(t) - x \geq 0$, and using (3.5), $w(c(t)) \geq w(c)$. Thus, $x(t) \in h(y(t),\, y(t+1))$, and $w(c(t)) = u(y(t),\, y(t+1))$.

Let $(a,b) \in \Omega$. Then defining $x \in h(a,b)$, and $c = a - x$, we have $(x,b) \in \Omega$ and $c \geq 0$, so by (3.59),

$$\begin{aligned} &\delta^t u(y(t),\ y(t+1)) + p(t+1)y(t+1) - p(t)y(t) \\ \geq\ &\delta^t u(a,b) + p(t+1)b - p(t)a \quad \text{ for all } (a,b) \in \Omega \end{aligned}$$

Thus, $\{y(t),\, p(t)\}_0^\infty$ is a competitive program from y, satisfying the transversality condition. Hence, by Theorem 3.3.1, $\{y(t)\}_0^\infty$ is an optimal program from y. Since we have already checked that $x(t) \in h(y(t),\, y(t+1))$, we can conclude that $\{x(t),\, y(t)\}_0^\infty$ is an optimal plan from y.

Proposition 3.7.2. *Suppose* $\{x(t),\, y(t)\}_0^\infty$ *is an optimal plan from* $y \in \mathbb{R}^n_{++}$. *Suppose, also, that there is some sufficient vector in* $\mathbb{R}^n_+$. *Then, there is a sequence* $\{p(t)\}_0^\infty$ *with* $p(t) \in \mathbb{R}^n_+$ *for* $t \geq 0$, *such that*

(i) $\{x(t),\, y(t),\, p(t)\}_0^\infty$ *is a competitive plan;*

(ii) *For all* $y \in \mathbb{R}^n_+$, *and* $t \geq 0$,

$$\delta^t V(y(t)) - p(t)y(t) \geq \delta^t V(y) - p(t)y \tag{3.60}$$

and

$$\lim_{t\to\infty} p(t)\, y(t) = 0. \tag{3.61}$$

Proof. Since $\{x(t),\, y(t)\}_0^\infty$ is an optimal plan from y, we have $x(t) \in h(y(t),\, y(t+1))$, and $\{y(t)\}_0^\infty$ is an optimal program from y. Hence, by Theorem 3.3.2, there is a sequence $\{p(t)\}_0^\infty$ such that $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, $\{y(t),\, p(t)\}_0^\infty$ is a competitive program from y, and (3.60), (3.61) hold. It remains to verify (i). This is accomplished by showing that for each $t \geq 0$, the price vector $p(t)$ provides the appropriate price support for both the consumption decision and the production decision.

Given any t, define $\theta_t(c) = \delta^t w(c) - p(t)c$ for all $c \in \mathbb{R}^n_+$, and $\pi_t(x,y) = p(t+1)y - p(t)x$ for all $(x,y) \in \Omega$.

Next, given t, we define the following two sets:

$A(t) = \{\alpha : \text{there exists } c \geq 0, \text{ satisfying } \theta_t(c) - \theta_t(c(t)) > \alpha\}$

$B(t) = \{\alpha : \text{there exists } (x, y) \in \Omega \text{ satisfying } \pi_t(x, y) - \pi_t(x(t), y(t+1)) > -\alpha\}$

We claim that (for each t),

$$A(t) \text{ and } B(t) \text{ are disjoint} \tag{3.62}$$

If (3.62) does not hold (for some t), there is some α which belongs to both $A(t)$ and $B(t)$. Then, there is $(x, y) \in \Omega$ and $c \geq 0$, such that $\theta_t(c) - \theta_t(c(t)) > \alpha$, and $\pi_t(x, y) - \pi_t(x(t),\, y(t+1)) > -\alpha$. Thus, we get:

$$\delta^t w(c) + p(t+1)y - p(t)(x+c) > \delta^t w(c(t)) + p(t+1)y(t+1) - p(t)y(t)$$

Defining $a = (x+c)$, we have $(a, y) \in \Omega$, and $u(a, y) \geq w(a-x) = w(c)$. Thus, $\delta^t u(a, y) + p(t+1)y - p(t)a \geq \delta^t w(c) + p(t+1)y - p(t)(x+c)$. Also, since $x(t) \in h(y(t),\, y(t+1))$, we have $w(c(t)) = w(y(t) - x(t)) = u(y(t),\, y(t+1))$. Hence,

$$\begin{aligned} \delta^t u(a, y) + p(t+1)y - p(t)a \quad > \quad & \delta^t u(y(t),\ y(t+1)) \\ & + p(t+1)y(t+1) - p(t)y(t) \end{aligned}$$

which contradicts the fact that $\{y(t),\, p(t)\}_0^\infty$ is a competitive program from y. This establishes our claim (3.62).

Next, we note that, by definition of the sets $A(t)$ and $B(t)$,

$$\text{(a) If } \alpha < 0, \text{ then } \alpha \in A(t), \text{ (b) If } \alpha > 0, \text{ then } \alpha \in B(t) \tag{3.63}$$

Now suppose there is some $c \in \mathbb{R}_+^n$, such that $\theta_t(c) > \theta_t(c(t))$. Then by defining $\alpha = \frac{1}{2}[\theta_t(c) - \theta_t(c(t))]$, we have $\alpha > 0$, and $\alpha \in A(t)$. By (3.63), $\alpha \in B(t)$, which contradicts (3.62). Hence $\theta_t(c) \leq \theta_t(c(t))$ for all $c \in \mathbb{R}_+^n$, which is (3.56).

Suppose there is some $(x, y) \in \Omega$ such that $\pi_t(x, y) > \pi_t(x(t),\, y(t+1))$. Then by defining $\alpha = -\frac{1}{2}[\pi_t(x, y) - \pi_t(x(t),\, y(t+1))]$, we note that $(-\alpha) = \frac{1}{2}[\pi_t(x, y) - \pi_t(x(t),\, y(t+1))]$, so $\alpha \in B(t)$, and $\alpha < 0$. By (3.63), $\alpha \in A(t)$, which contradicts (3.62). Thus $\pi_t(x, y) \leq \pi_t(x(t),\, y(t+1))$ for all $(x, y) \in \Omega$, which is (3.57).

We have now shown that $\{x(t),\, y(t),\, p(t)\}_0^\infty$ is a competitive plan from y so that (i) holds. This completes the proof of the proposition.

3.7.4 Existence of a Stationary Optimal Output

The existence of a modified golden-rule equilibrium and a non-trivial stationary optimal stock have been obtained in the literature by Peleg and Ryder (1974) by using duality theory. This result can be obtained as a special case of Theorem 3.4.2 and Corrollary 3.4.1, which we have established for the general framework of Section 3.2.

Call the technology set $\delta - productive$ if there exists $(\bar{x}, \bar{y})$ in Ω such that $\delta\bar{y} >> \bar{x}$. Note that if Ω is δ-productive, then with the definition of u given in Section 3.7.1, and assumptions (B.4) and (B.6), (Ω, u, δ) is $\delta u - productive$. For $(\delta\bar{y}, \bar{y})$ is clearly in Ω by (B.4), and $\bar{x}$ is in $g(\delta\bar{y}, \bar{y})$. So $u(\delta\bar{y}, \bar{y}) \geq w(\delta\bar{y} - \bar{x}) > w(0) = u(0, 0)$.

Proposition 3.7.3. *If Ω is δ-productive, there is a triple $(\hat{x}, \hat{y}, \hat{p})$ such that $(\hat{x}, \hat{y}, \hat{p})$ is a modified golden-rule equilibrium. Furthermore, $\hat{y}$ is a non-trivial stationary optimal output.*

Proof. Since Ω is δ-productive, it is also δu-productive. So, using Theorem 3.4.2, there is a pair $(\hat{y}, \hat{p})$ such that $(\hat{y}, \hat{p})$ is a modified golden-rule and $\hat{y}$ is a non-trivial stationary optimal stock. That is, $(\hat{y}, \hat{y}) \in \Omega$, $\hat{p} \in \mathbb{R}^n_+$, and for all $(a, b) \in \Omega$,

$$u(\hat{y}, \hat{y}) + \delta\hat{p}\hat{y} - \hat{p}\hat{y} \geq u(a, b) + \delta\,\hat{p}\hat{b} - p\,a \tag{3.64}$$

Let $\hat{x}$ be an element of $h(\hat{y}, \hat{y})$. Then, $(\hat{x}, \hat{y}) \in \Omega$, and denoting $(\hat{y} - \hat{x})$ by $\hat{c}$, we have $\hat{c} \geq 0$ and $w(\hat{c}) = u(\hat{y}, \hat{y})$.

Define $\theta(c) \equiv w(c) - \hat{p}c$ for all $c \in \mathbb{R}^n_+$, and $\pi(x, y) \equiv \delta\hat{p}y - \hat{p}x$ for all $(x, y) \in \Omega$. Now, following the method of proof in Proposition 3.7.2, one can establish that $\theta(c) \leq \theta(\hat{c})$ for all $c \in \mathbb{R}^n_+$, and $\pi(x, y) \leq \pi(\hat{x}, \hat{y})$ for all $(x, y) \in \Omega$. Hence, $(\hat{x}, \hat{y}, \hat{p})$ is a modified golden-rule equilibrium.

Using Proposition 3.7.1, $\{\hat{y}\}_0^\infty$ is a stationary optimal program from $\hat{y}$. Since $\hat{y}$ is a non-trivial stationary optimal stock, it is also a non-trivial stationary optimal output.

3.8 Weitzman's Theorem on the NNP

Weitzman (1976) showed, in a continuous time optimal growth model, that at each instant of time, the present value of the net national product at that instant of time (evaluated at the current supporting prices) equals the maximum discounted sum of utilities the economy is capable of achieving from that time onwards. This is an interesting economic interpretation of the Bellman equation of dynamic programming (in continuous time). We provide here a discrete time analog of Weitzman's observation which, although it does not have the force of his result (discrete-time does not allow us to conclude equality between the two relevant magnitudes), might be of interest. Our approach to this result is an application of the methods of duality theory, discussed in Sections 3.3 and 3.7 above.

Our framework of analysis is the "consumption model" described in Section 3.7. If $\{x(t), y(t), p(t)\}$ is a competitive plan, then the *current value* price sequence $\{q(t)\}$, associated with the plan, is defined by: $q(t) = p(t)/\delta^t$ for $t \geq 0$.

Theorem 3.8.1. *If $\{x(t), y(t)\}_0^\infty$ is an optimal plan from a sufficient vector $y \in \mathbb{R}^n_{++}$, and $\{p(t)\}_0^\infty$ is a sequence with $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$ such that*

$\{x(t), y(t), p(t)\}$ is a competitive plan satisfying (3.60) and (3.61). Then, for each $s \geq 1$

$$V(y(s)) \geq \frac{w(c(s)) + q(s)(x(s) - x(s-1))}{(1-\delta)} \tag{3.65}$$

and

$$V(y(s)) \leq \frac{w(c(s)) + q(s-1)(x(s) - x(s-1))}{(1-\delta)} \tag{3.66}$$

Proof. Pick any $s \geq 1$, and use $t = s+1$ and $y = y(s)$ in (3.60) to get

$$V(y(s+1)) - V(y(s)) \geq q(s+1)(y(s+1) - y(s)) \tag{3.67}$$

Use $t = s$ and $(x(s-1), y(s)) \in \Omega$ in (3.57) to get

$$q(s+1)(y(s+1) - y(s)) \geq (q(s)/\delta)(x(s) - x(s-1)) \tag{3.68}$$

Using (3.67) and (3.68),

$$V(y(s+1)) \geq V(y(s)) + (q(s)/\delta)(x(s) - x(s-1)) \tag{3.69}$$

By the principle of optimality, we also have

$$V(y(s)) = w(c(s)) + \delta V(y(s+1)) \tag{3.70}$$

So, using (3.69) in (3.70), we get

$$V(y(s)) \geq w(c(s)) + \delta V(y(s)) + q(s)(x(s) - x(s-1))$$

Transposing terms,

$$(1-\delta)V(y(s)) \geq w(c(s)) + q(s)(x(s) - x(s-1))$$

which yields (3.65).

Following an entirely analogous method, we can use $t = s$ and $y = y(s+1)$ in (3.60) to get

$$V(y(s)) - V(y(s+1)) \geq q(s)(y(s) - y(s+1)) \tag{3.71}$$

Use $t = s-1$ and $(x(s),\ y(s+1)) \in \Omega$ in (3.57) to get

$$q(s)(y(s) - y(s+1)) \geq (q(s-1)/\delta)(x(s-1) - x(s)) \tag{3.72}$$

Using (3.71) and (3.72),

$$V(y(s)) - V(y(s+1)) \geq (q(s-1)/\delta)(x(s-1) - x(s)) \tag{3.73}$$

Transposing terms,

$$V(y(s+1)) \leq V(y(s)) + (q(s-1)/\delta)(x(s) - x(s-1)) \tag{3.74}$$

Using the principle of optimality (3.70), we have:

$$V(y(s)) \leq w(c(s)) + \delta V(y(s)) + q(s-1)(x(s) - x(s-1))$$

Transposing terms,

$$(1-\delta)V(y(s)) \leq w(c(s)) + q(s-1)(x(s) - x(s-1))$$

which yields (3.66).

Remark:

(i) Note that the hypothesis of the Theorem can be seen to be non-vacuous by an appeal to Proposition 3.7.2 in Section 3.7. That is, given an optimal plan $\{x(t), y(t)\}_0^\infty$ from a sufficient vector $y \in \mathbb{R}^n_{++}$, there exists a price sequence $\{p(t)\}_0^\infty$ with $p(t) \in \mathbb{R}^n_+$ for $t \geq 0$, such that $\{x(t), y(t), p(t)\}$ is a competitive plan satisfying (3.60) and (3.61).

(ii) Our theorem indicates that the maximum discounted sum of utilities achievable from time s onwards [that is, $V(y(s))$] is trapped between two magnitudes, each of which has some claim to be interpreted as the present-value of the net national product in time period s. The difference between the two magnitudes is the "current" price ($q(s)$ or $q(s-1)$) used to evaluate investment $(x(s) - x(s-1))$ during the time period s.

3.9 Bibliographic Notes

Section 3.2:
The general framework described in this section was introduced into the literature by Gale (1967) and McKenzie (1968) in their contributions on optimal growth when future utilities are *undiscounted*. The framework has great flexibility, and a variety of intertemporal allocation problems can be reduced to this framework. The well-known model, in which utility is derived from consumption alone, is discussed in Section 3.7 as an illustration of this observation. For other intertemporal allocation problems, see the exposition in Mitra (2000).

Section 3.3:
The section describes the basic price characterization results in the discounted case, following the approach of Weitzman (1970). The approach can be adapted to the undiscounted case as well; for this, see Peleg and Zilcha (1977) and McKenzie (1986).

Theorem 3.3.1 does not require convex structures, but under non-convexities it turns out to be not a useful tool for showing that a candidate program is optimal, since in general one will not be able to obtain a price sequence at which the program will satisfy the competitive conditions. When an optimal program is interior, and the utility function is differentiable in the interior of the transition possibility set, a necessary condition of optimality is the Ramsey-Euler equation. In this case, it can also be shown that the optimal program satisfies a suitable transversality condition. However, the Ramsey-Euler conditions together with this transversality condition is not sufficient for optimality in non-convex models.

The results of this section can be generalized to a setting involving changing technology and tastes. For the general theory, see McKenzie (1974); for applications to an aggregative model, see Mitra and Zilcha (1981).

Section 3.4:
The approach, consisting of establishing the existence of a discounted golden-rule, as a step to establishing the existence of a non-trivial stationary optimal stock, is due to Flynn (1980) and McKenzie (1982). Khan and Mitra (1986) showed that this could be accomplished when continuity of the utility function is replaced by upper semicontinuity, thereby making the result more widely applicable. They also showed that duality methods could be completely dispensed with in establishing that the discounted golden-rule stock is a non-trivial stationary optimal stock. The approach of Peleg and Ryder (1974) is somewhat similar, but their method applies only to the "consumption model". The dynamic programming approach of Sutherland (1970) runs into the problem that the stationary optimal stock obtained by the fixed point argument can be trivial, and there is no obvious way to ensure non-triviality of the fixed point, even when the economy is $\delta u - productive$.

Analogous results for the undiscounted case (involving the notion of a golden-rule) are contained in Gale (1967), McKenzie (1968), Brock (1970) and Peleg (1973). However, a major difference is that the existence of a golden-rule can be shown without any use of fixed point methods.

Section 3.5:
The idea of replacing the transversality condition by a period-by-period condition was first proposed by Brock and Majumdar (1988), who established the appropriate result in the undiscounted case, for the "consumption model". For application of the same principle in other settings, see the collection of papers, edited by Majumdar (1992).

Section 3.6:
The reachability condition proposed here is weak. Other related conditions, such as local expandability and local contractability, proposed by Kurz and Starrett (1970), are more restrictive. The example of the simple Leontief model (as described by Gale (1960)) is an instance where the reachability condition can be checked quite easily, but both local expandability and local contractability fail.

Section 3.7:
Viewing the "consumption model" as a special case of the general framework of Section 3.2 has the advantage that many duality results, developed for the consumption model (see, especially, Peleg and Ryder (1972, 1974)), can be obtained by an alternative and simpler route, and the assumptions needed for either approach to work can thereby be compared. For a more complete discussion, see Dasgupta and Mitra (1990).

Section 3.8:
Weitzman's Rule is almost exclusively discussed in the literature in the context of continuous models. The discrete-time analog presented here indicates that the argument involved is quite elementary, and is a good illustration of the essential simplicity of duality methods.

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